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Surfaces in $\mathbb{R}^3$ with prescribed curvature

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Abstract

We find conditions on the Gauss map and the Gauss curvature to describe a surface S in $\mathbb{R}^3$. Given the curvature of S (e.g. as a function of the asymptotic coordinates) these conditions are equivalent to the reduced σ-model equations. We interpret the Kerr solution of the Ernst equation as a surface in $\mathbb{R}^3$.

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1. Introduction

In this paper we study two-dimensional immersed submanifolds (surfaces) S of the flat space $\mathbb{R}^3$ with the Euclidean or Minkowskian metric. The first fundamental form of S is given by

$$g_{\rm I} = g_{ij}\, dx^i\, dx^j, \quad g_{ij} = \mathrm{diag}(1, 1, \epsilon),\ \epsilon = \pm 1\,, \tag{1}$$

where x^i $(i = 1, 2, 3)$ are functions of coordinates on S (pullbacks of the Cartesian coordinates of $\mathbb{R}^3$ under the immersion). We are especially interested in surfaces with a prescribed Gauss (Ricci scalar) curvature. We assume that the curvature is given in terms of coordinates, which fix the second fundamental form $g_{\rm II}$ of S up to a conformal factor (see (3) and (13)). Such surfaces were already investigated by Bianchi [4]. Recently, they were studied by Levi and Sym [13] (see also [5,9,10]), who considered equations describing hyperbolic surfaces in the three-dimensional Euclidean space ($\epsilon = 1$ in (1)) from the point of view of their complete integrability. This paper is an extended version of the unpublished work of the author [15].

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Levi and Sym [13] use the following parametrization of the first and the second fundamental forms of S:

$$g_{\mathrm{I}} = \rho^2(a^2\,\mathrm{d}x^2 + 2ab\cos\varphi\,\mathrm{d}x\,\mathrm{d}t + b^2\,\mathrm{d}t^2)\,, \tag{2}$$

$$g_{\mathrm{II}} = 2\rho ab\sin\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{3}$$

The functions x and t are called the asymptotic coordinates of S and $K = -\rho^{-2}$ is the Gauss curvature of S. Conditions for the functions a, b, φ, ρ of x and t to define a surface are given by the Gauss–Codazzi equations [7]

$$\varphi_{,xt} + \left(\frac{b\rho_{,x}}{2a\rho}\sin\varphi\right)_{,x} + \left(\frac{a\rho_{,t}}{2b\rho}\sin\varphi\right)_{,t} - ab\sin\varphi = 0\,, \tag{4}$$

$$2\rho a_{,t} + a\rho_{,t} - b\rho_{,x}\cos\varphi = 0\,, \tag{5}$$

$$2\rho b_{,x} + b\rho_{,x} - a\rho_{,t}\cos\varphi = 0\,. \tag{6}$$

It was assumed in [13] that the function $\rho(x,t)$ is given. In this case Eqs. (4)–(6) yield three equations for three unknown functions a, b, φ. Levi and Sym raised the question when this system is completely integrable. They showed that Eqs. (4)–(6) coincide with integrability conditions of the following linear system of equations (the Gauss–Weingarten equations) for a wave function ψ:

$$\psi_{,x} = \left[\tfrac{1}{2}\mathrm{i}a\sigma_3 + \mathrm{i}\left(\tfrac{1}{2}\varphi_{,x} + \tfrac{1}{4}(a\rho_{,t}/b\rho)\sin\varphi\right)\sigma_2\right]\psi\,, \tag{7}$$

$$\psi_{,t} = \left(-\tfrac{1}{2}\mathrm{i}b\cos\varphi\sigma_3 - \tfrac{1}{4}\mathrm{i}(b\rho_{,x}/a\rho)\sin\varphi\sigma_2 + \tfrac{1}{2}\mathrm{i}b\sin\varphi\sigma_1\right)\psi\,, \tag{8}$$

where $\psi(x,t)$ is a 2×2 invertible matrix and σ_i are the Pauli matrices. In general, one cannot introduce a spectral parameter λ into Eqs. (7) and (8). This is possible when

$$\rho(x,t) = \rho_1(x) + \rho_2(t)\,. \tag{9}$$

A resulting spectral problem admits the Zakharov–Shabat dressing method (called the Darboux transformation in [13]). Thus, it is justified to call Eqs. (4)–(6) completely integrable when ρ is given by (9).

In this paper, we present another description of surfaces in $\mathbb{R}^3$. We concentrate on properties of the normal (Gauss) mapping. We find conditions on the normal unit vector field and the Gauss curvature to describe a surface in $\mathbb{R}^3$ (Section 2). For $\epsilon = 1$ and an indefinite second fundamental form g_{II} they are equivalent to Eqs. (4)–(6). Given the Gauss curvature (in terms of coordinates normalizing g_{II}) our equations coincide with a two-dimensional symmetry reduction of the three-dimensional σ-model equations . Assumption (9) leads to known completely integrable models. In particular, if the functions ρ_1 and ρ_2 are both nonconstant, one obtains the Ernst equation (or its Euclidean analogue) describing stationary axisymmetric gravitational fields in the Einstein theory. In Section 3 we construct a surface in $\mathbb{R}^3$ corresponding to the Kerr solution, which plays a crucial role in the theory of black holes.

We use the index notation of general relativity. Repeated indices are summed over their range. The matrix g^{ij} is the inverse of g_{ij}. Indices are raised and lowered by means of g^{ij} and g_{ij}, $v^i = g^{ij}v_j$, $v_i = g_{ij}v^j$. The relativistic square of a vector (or covector) v is defined by $v^2 = v^i v_i$. Given a metric tensor one can define the Levi-Civita completely antisymmetric pseudotensor. In the case of $\mathbb{R}^3$ and the Cartesian coordinates we denote it by ϵ^{ijk}. All functions are assumed to be at least twice continuously differentiable.

2. Conditions on the normal mapping

The main results of this paper are given by the following propositions.

Proposition 1. *Let S be an immersed surface in the flat space $\mathbb{R}^3$. Assume that the first and the second fundamental forms g_{I}, g_{II} of S are nondegenerate at each point. Then*

$$\mathrm{d}x^i = \rho\epsilon^{ijk}n_j\,{}^*\mathrm{d}n_k\ , \tag{10}$$

where n_i is the unit vector field normal to S, $n^2 = \pm 1$, the star denotes the Hodge dualization with respect to g_{II} and $|\rho| = |K|^{-1/2}$, K being the Gauss curvature of S.

Proof. Since g_{I} is nondegenerate the normal vector n is not null and we can assume $n^2 = \pm 1$. The first and the second fundamental forms of S are given by

$$g_{\mathrm{I}} = \mathrm{d}x^i\,\mathrm{d}x_i \tag{11}$$

and

$$g_{\mathrm{II}} = \mathrm{d}x^i\,\mathrm{d}n_i\ . \tag{12}$$

If g_{II} is nondegenerate one can introduce local coordinates ξ^a, $a = 1, 2$, on S such that

$$g_{\mathrm{II}} = f(\mathrm{d}\xi^1\,\mathrm{d}\xi^1 + \epsilon'\,\mathrm{d}\xi^2\,\mathrm{d}\xi^2)\ ,\quad f \neq 0\ ,\ \epsilon' = \pm 1\ . \tag{13}$$

It follows from (12) and (13) that

$$x^i_{,1}n_{i,2} + x^i_{,2}n_{i,1} = 0\ , \tag{14}$$

$$f = x^i_{,1}n_{i,1} = \epsilon' x^i_{,2}n_{i,2}\ , \tag{15}$$

where $x_{,a}(n_{,a})$ denotes the derivative of $x(n)$ with respect to ξ^a. Moreover, since n is orthogonal to S,

$$x^i_{,a}n_i = 0 \tag{16}$$

and

$$-x^i_{,1,2}n_i = x^i_{,1}n_{i,2} = x^i_{,2}n_{i,1}\ . \tag{17}$$

Eqs. (14) and (17) yield

$$x^i_{,1}n_{i,2} = x^i_{,2}n_{i,1} = 0\ . \tag{18}$$

Vectors $n, n_{,1}, n_{,2}$ have to be independent since $n^i_{,a} n_i = 0$ and a proportionality of $n_{,1}$ and $n_{,2}$ would imply $f = 0 = g_{\mathrm{II}}$ in virtue of (15) and (18). This fact is equivalent to the condition

$$\epsilon^{ijk} n_i n_{j,1} n_{k,2} \neq 0 . \tag{19}$$

Due to (19) Eqs. (16) and (18) yield

$$x^i_{,1} = -\rho \epsilon^{ijk} n_j n_{k,2} , \tag{20}$$

$$x^i_{,2} = \rho' \epsilon^{ijk} n_j n_{k,1} , \tag{21}$$

where ρ, ρ' are functions of ξ^a. It follows from (15) that

$$\rho' = \epsilon' \rho . \tag{22}$$

Eq. (10) is a direct consequence of (20)–(22) since $^*\mathrm{d}\xi^1 = \mathrm{d}\xi^2$, $^*\mathrm{d}\xi^2 = -\epsilon' \mathrm{d}\xi^1$ for the metric (13) (independently of f). Substituting (20)–(22) into (11) and (12) yields

$$g_{\mathrm{I}} = \epsilon n^2 \rho^2 (n_{i,2} n^i_{,2} \mathrm{d}\xi^1 \mathrm{d}\xi^1 - 2\epsilon' n_{i,1} n^i_{,2} \mathrm{d}\xi^1 \mathrm{d}\xi^2 + n_{i,1} n^i_{,1} \mathrm{d}\xi^2 \mathrm{d}\xi^2), \tag{23}$$

$$g_{\mathrm{II}} = \rho \epsilon^{ijk} n_i n_{j,1} n_{k,2} (\mathrm{d}\xi^1 \mathrm{d}\xi^1 + \epsilon' \mathrm{d}\xi^2 \mathrm{d}\xi^2) . \tag{24}$$

The Gauss curvature of S is equal to the Ricci scalar curvature. It can be defined as $K = \det g_{\mathrm{II}} / \det g_{\mathrm{I}}$ [7]. It follows from (23) and (24) that

$$K = \epsilon \epsilon' n^2 \rho^{-2} . \tag{25}$$

Note that $\epsilon \epsilon' n^2 = \pm 1$ and that Eq. (10) does not depend on the choice of coordinates on S. □

Proposition 2. *Let S be a two-dimensional differential manifold with metric g' given up to a conformal factor. Let functions ρ, n_i on S satisfy the following equation*

$$\epsilon^{ijk} n_j \, \mathrm{d}(\rho \, {}^*\mathrm{d} n_k) = 0 \tag{26}$$

together with the conditions

$$g^{ij} n_i n_j = \pm 1 , \tag{27}$$

$$\rho \epsilon^{ijk} \mathrm{d} n_j \, {}^*\mathrm{d} n_k \neq 0 , \tag{28}$$

where g^{ij} is given by (1) *and the Hodge star corresponds to g'. Then, locally, S can be considered as a submanifold of $\mathbb{R}^3$ such that n is the normal vector field, the Gauss curvature of S is given by* (25) *and the second fundamental form of S is proportional to g'.*

Proof. Eq. (26) implies (locally) the existence of functions x^i on S such that (10) is satisfied. These functions define a local embedding (an immersion if x^i exists globally) of S into $\mathbb{R}^3$. Let ξ^a, $a = 1, 2$, be coordinates of S such that

$$g' \sim (\mathrm{d}\xi^1 \mathrm{d}\xi^1 + \epsilon' \mathrm{d}\xi^2 \mathrm{d}\xi^2) , \quad \epsilon' = \pm 1 . \tag{29}$$

In these coordinates (10) is equivalent to Eqs. (20)–(22). Substituting these equations into the definitions (11), (12) yields expressions (23) and (24) for the fundamental forms. Condition (28) implies (19) and $\rho \neq 0$, hence g_{I} and g_{II} are nondegenerate. It follows from (23), (24), (29) that $g_{\mathrm{II}} \sim g'$ and the Gauss curvature is given by (25). □

Remark 1. In the following we will assume that Eq. (26) is always accompanied by conditions (27) and (28).

Eq. (26) becomes an equation for the Gauss map if ρ (or the Gauss curvature) is given as a function of coordinates on S. For $\rho = 1$, Eq. (26) describes a harmonic map from S with the metric g' (or g_{II}) into the two-dimensional sphere or pseudosphere. This equation is often called the σ-model equation. For $\rho \neq$ const., Eq. (26) can be also considered as a harmonic map equation provided the domain of n is extended trivially to the manifold $\tilde{S} = \mathbb{R} \times S$ with the metric

$$\tilde{g} = ag' + \rho^2(\mathrm{d}\phi + \omega)^2 , \quad a_{,\phi} = \omega_{,\phi} = 0. \tag{30}$$

Here φ is a coordinate of $\mathbb{R}$, a is a nonvanishing function on S and ω is a 1-form on S. Metric (30) is preserved by translations in ϕ. And conversely, if a three-dimensional metric admits a Killing vector it can be transformed into the form (30). Thus, in general, Eq. (26) coincides with a reduction of a three-dimensional σ-model equation by means of a symmetry. The function ρ is equal to the length of the Killing vector.

For $\epsilon = -\epsilon' = 1$, Eq. (26) and Eqs. (4)–(6) describe the same geometrical situation. Hence they must be equivalent (see also [9] for a proof of this fact). The function ρ is the same in both approaches, but the variables a, b, φ are related to the components of n by a nonpoint transformation. A transformation between coordinates x, t and ξ^a is given by

$$x = \xi^1 + \xi^2, \qquad t = \xi^1 - \xi^2. \tag{31}$$

Comparing expressions (2) and (23) for g_{I} yields functions a, b, φ in terms of n^i,

$$a = (n^i_{,x} n_{i,x})^{1/2}, \qquad b = (n^i_{,t} n_{i,t})^{1/2}, \qquad ab\cos\varphi = -n^i_{,x} n_{i,t} . \tag{32}$$

It follows from Proposition 2 that so defined functions a, b, φ correspond to some surface in $\mathbb{R}^3$ if ρ and n satisfy Eq. (26). Hence the Gauss–Codazzi Eqs. (4)–(6) must be necessarily satisfied (this fact was confirmed by Levi [12] with the use of a symbolic computation package). In order to pass from a, b, φ to n one can use Eqs. (7) and (8). Let ψ be a solution of these equations taking values in SU(2):

$$\psi^\dagger = \psi^{-1}. \tag{33}$$

(Note that condition (33) is preserved by Eqs. (7) and (8), hence it is sufficient to impose it only at a point.) A direct calculation shows, in virtue of (4)–(8), that

$$(\rho J^{-1} J_{,t})_{,x} + (\rho J^{-1} J_{,x})_{,t} = 0 , \tag{34}$$

where $J = \sqrt{-1}\psi^{-1}\sigma_2\psi$ (hence $J \in \mathrm{SU}(2)$ and $\mathrm{Tr}\, J = 0$). Eq. (34) is equivalent to (26) modulo transformation (31) and the identification

$$\psi^{-1}\sigma_2\psi = n^i\sigma_i \,. \tag{35}$$

It follows that functions n^i given by (35) define a surface S in $\mathbb{R}^3$ with the curvature ρ. One can recover the original functions a, b, φ by means of (32). Hence, the surface S coincides with that related to a, b, φ in the approach of Levi and Sym. (More precisely, a, b, φ define a surface up to a rigid motion [13]. This fact agrees with the ambiguity in the choice of ψ that gives rise to n.)

Remark 2. Given a surface S (hence the functions x^i, n_i, ρ) there is a dual surface S' for which the roles of x^i and n^i interchange in the following sense

$$x'^i = \frac{n^i}{x^k n_k}\,, \qquad n'^i = \frac{x^i}{\sqrt{x^k x_k}}\,, \qquad \rho\rho' = (n^i n'_i)^{-1}\,. \tag{36}$$

It can be verified by a direct calculation that the primed variables satisfy Eq. (10) if the unprimed variables do. Transformation (36) does not preserve the property (39) (see Section 3). One can obtain an infinite series of surfaces by applying (36) alternatively with translations of x by a constant vector.

Remark 3. The Gauss curvature K of S is well defined if g_{I} is not degenerate. In this case the metric g_{II} is nondegenerate iff $K \neq 0$. When $K = 0$ everywhere then there are local coordinates y, z on S such that $g_{\mathrm{II}} \sim \mathrm{d}z^2$. Following the proof of Proposition 1 one can easily show that then S is either a plane or it is given locally by

$$x^i = y\epsilon^{ijk} n_j(z) n_{k,z} + a^i(z), \quad n^2 = \pm 1, \; n_i a^i_{\ ,z} = 0, \tag{37}$$

where n_i, a^i depend only on z. A surface defined by (37) is formed by straight lines emanating in a peculiar way from the curve $x^i = a^i(z)$. If a surface S is everywhere null, $n^2 = 0$, then there is no notion of the Gauss curvature. In this case $g_{\mathrm{I}} \sim g_{\mathrm{II}} \sim \mathrm{d}z^2$ in some coordinates y, z. The surface is given locally by

$$x^i = yn^i(z) + a^i(z), \quad n^2 = n_i a^i_{\ ,z} = 0. \tag{38}$$

It follows from (38) that S is formed by null straight lines emanating orthogonally from a non-timelike curve in a three-dimensional Minkowski space.

3. Surface related to the Kerr metric

Eq. (26) has special properties when ρ is a harmonic function on S with respect to the metric g' (or g_{II})

$$\mathrm{d}^*\mathrm{d}\rho = 0. \tag{39}$$

Then either

$$\rho = \text{const.} \tag{40}$$

or one can adapt the coordinates ξ^a (see (13)) to obtain

$$\rho = \xi^1 \tag{41}$$

or

$$\rho = \xi^1 + \xi^2 \quad (\text{if } \epsilon' = -1). \tag{42}$$

Pohlmeyer [14] showed that for $\rho = \text{const.}$, Eq. (26) is equivalent to the sine-Gordon equation (the latter equation follows immediately from (4)). Hence it is clear that in this case Eq. (26) is completely integrable. A direct proof of this property and a construction of solutions via the dressing method of Zakharov and Shabat was given by Zakharov and Mikhailov [18]. Belinski and Zakharov [1,2] applied successfully a similar method when $\rho \neq \text{const.}$ and (39) is satisfied. In the case of (41), Eq. (26) is equivalent to the Ernst equation (or its analogue) of general relativity. The Ernst equation admits also other solution generating methods, which were found in late seventies (see e.g. [11]). Concluding, Eq. (26) is completely integrable whenever ρ satisfies (39). This fact agrees with the results of Levi and Sym [13] on Eqs. (4)–(6). It also agrees with the observation that Eq. (26), with ρ given by any of the formulas (40)–(42), can be considered as a symmetry reduction of the self-dual Yang–Mills equations (SDYM) (see e.g. [16,17]), which are completely integrable. In all these cases a corresponding spectral problem can be obtained from that for the SDYM equations [16].

The Ernst equation follows from the Einstein equations under the assumption that a metric tensor solving the Einstein equations is stationary and axially symmetric and the two symmetries commute. It is equivalent to Eq. (26) provided

$$\epsilon = -\epsilon' = n^2 = -1 \tag{43}$$

and $\rho \neq \text{const.}$ satisfies (39). Hence, n corresponds to a spacelike surface in the three-dimensional Minkowski space. Under these assumptions $g_{II} \sim g'$, where

$$g' = (\mathrm{d}\rho^2 + \mathrm{d}z^2) \tag{44}$$

and ρ, z are coordinates of S (see (13) and (41), $z = \xi^2$). By means of the stereographic projection the vector n can be parametrized in the following way:

$$n_2 + i n_1 = \frac{2\zeta}{\zeta\bar{\zeta} - 1}, \qquad n_3 = \frac{\zeta\bar{\zeta} + 1}{\zeta\bar{\zeta} - 1}, \tag{45}$$

where ζ is a complex variable. In terms of ζ Eq. (26) reads

$$(\zeta\bar{\zeta} - 1)\, \mathrm{d}(\rho\, {}^*\mathrm{d}\zeta) = 2\rho\bar{\zeta}\, \mathrm{d}\zeta \wedge {}^*\mathrm{d}\zeta\,, \tag{46}$$

where the Hodge dual is taken with respect to g'. Substituting $\zeta = (1 - E)/(1 + E)$ in (46) yields the Ernst equation

$$\operatorname{Re} E\, \mathrm{d}(\rho\, {}^*\mathrm{d}E) = \rho\, \mathrm{d}E \wedge {}^*\mathrm{d}E\,. \tag{47}$$

Condition (28) is equivalent to

$$\mathrm{d}E \wedge \mathrm{d}\bar{E} \neq 0\,. \tag{48}$$

It excludes solutions of the Ernst equation belonging to the Weyl class ($E = \bar{E}$ in this class). Eq. (47) admits the point symmetry group SL(2, $\mathbb{R}$), which corresponds to the Lorentz invariance of Eq. (26).

One of the most important solutions of the Ernst equation is that related to the Kerr metric [11,8], which represents a gravitational field of a rotating black hole. To describe this it is convenient to use the so-called prolate spheroidal coordinates w, y, where $w > 1$, $|y| < 1$ in the region outside the horizon and the axis of rotation. In these coordinates

$$\rho = \sigma\sqrt{(w^2-1)(1-y^2)}, \qquad \zeta = (pw - \mathrm{i}qy)^{-1}\,, \quad p^2+q^2=1\,, \quad pq \neq 0\,, \tag{49}$$

$$g_{\mathrm{II}} \sim \left(\frac{\mathrm{d}w^2}{w^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right), \tag{50}$$

where σ, p, q are constants. From (49) and (45) one obtains

$$n = h(2qy, 2pw, 1 + p^2w^2 + q^2y^2)\,, \tag{51}$$

where

$$h = (1 - p^2w^2 - q^2y^2)^{-1}. \tag{52}$$

Integrating (10) yields

$$x = -\sigma h(2py(w^2-1), 2qw(1-y^2), (p/q)\,(w^2-1) + (q/p)\,(1-y^2)) \tag{53}$$

modulo addition of a constant vector. Eq. (53) defines parametrically a part of a surface $\tilde{S}$ in $\mathbb{R}^3$ given by the equation

$$\frac{(x^1)^2}{x^3+c} + \frac{(x^2)^2}{x^3-c} = x^3 - b\,, \tag{54}$$

where $c = \sigma/pq$, $b = \sigma\,((q/p) - (p/q))$ (c can be normalized to value 1 by a scaling of x^i). The surface $\tilde{S}$ is spacelike everywhere except four straight lines given by $x^1 = \pm p(x^3 - c)$, $x^2 = \pm q(x^3 + c)$, where the normal vector becomes null ($n^2 = 0$). Also the parametrization (53) (with relaxed constraints on values of w and y) is valid apart these lines. Intersections of $\tilde{S}$ with planes $x^3 =$ const. are ellipses for $|x^3| > |c|$, hyperbolas when $|x^3| < |c|$ and they contract to an interval when $x^3 = \pm c$. For $|x^3| \to \infty$ the surface $\tilde{S}$ tends to a light cone, which corresponds to the ergosphere in the Kerr space–time. The fundamental forms of $\tilde{S}$ read as follows in the coordinates w, y:

$$g_{\mathrm{I}} = -4\sigma^2h^2[q^2(1-y^2)^2(\mathrm{d}w)^2 + p^2(w^2-1)^2(\mathrm{d}y)^2], \tag{55}$$

$$g_{\mathrm{II}} = 4c^{-1}\sigma^2h^2[(1-y^2)(\mathrm{d}w)^2 + (w^2-1)(\mathrm{d}y)^2]\,. \tag{56}$$

The form g_{II} is definite ($\epsilon' = 1$) in some domains of $\tilde{S}$ and it is indefinite ($\epsilon' = -1$) in others. Since both fundamental forms are diagonal in the coordinates w, y it follows that $\tilde{S}$ is an isothermic surface [3,6].

An Euclidean analogue ($\epsilon = 1$) of the surface (54) follows when x^1, x^2 are replaced by purely imaginary variables. For $c = 1$ one obtains

$$\frac{(x^1)^2}{x^3+1} + \frac{(x^2)^2}{x^3-1} = b - x^3 \, . \tag{57}$$

For $b > x^3 > 1$ the surface given by (57) is bounded. It can be completed to a closed set $\tilde{S}$ by attaching an interval connecting the points $x_\pm = (\pm\sqrt{(b-1)/2}, 0, 1)$. $\tilde{S}$ is smooth everywhere except the points $x_\pm$, in which n has no limit. Intersections of $\tilde{S}$ with planes $x^3 =$ const. are ellipses, which contract to a point when $x^3 \to b$ and to the interval when $x^3 \to 1$. The second fundamental form of S is degenerate on the interval.

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References

[1] V.A. Belinski and V.E. Zakharov, Integration of the Einstein equations by the inverse scattering problem technique and the calculation of the exact soliton solutions, Sov. Phys. JETP 48 (1978) 985; Zh. Eksp. Teor. Fiz. 75 (1978) 1955.

[2] V.A. Belinski and V.E. Zakharov, Stationary gravitational solutions with axial symmetry, Sov. Phys. JETP 50 (1979) 1; Zh. Eksp. Teor. Fiz. 77 (1979) 3.

[3] L. Bianchi, Complementi alle ricerche sulle superficie isoterme, Annali di Matematica, serie 3, XII (1906) 17.

[4] L. Bianchi, *Lezioni di Geometria Differenziale* (Spoerri, Pisa, 1922).

[5] A. Bobenko, Surfaces in terms of 2 × 2 matrices. Old and new integrable cases, in: *Harmonic Maps and Integrable Systems*, eds. A.P. Fordy and J.C. Woods, Aspects of Math. 23 (Vieweg, 1993).

[6] S.S. Chern, Deformation of surfaces preserving principal curvatures, in: *Differential Geometry and Complex Analysis*, eds. I. Chavel and H.M. Farkas (Springer, Berlin, 1985).

[7] M.P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Englewood Cliffs, NJ, 1981).

[8] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space–Time* (Cambridge University Press, Cambridge, 1973).

[9] D. Korotkin, On some integrable cases in surface theory, Hamburg University preprint No. 116 (1994).

[10] D. Korotkin and V. Reznik, Bianchi surfaces in R^3 and deformation of hyperelliptic curves, Matem. Zametki 52 (1992) 78 (in Russian).

[11] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact Solutions of Einstein's Field Equations* (VEB Deutscher Verlag, Berlin, 1980).

[12] D. Levi, private communication.

[13] D. Levi and A. Sym, Integrable systems describing surfaces of non-constant curvature, Phys. Lett. A 149 (1990) 381.

[14] K. Pohlmeyer, Integrable hamiltonian systems and interactions through quadratic constraints, Commun. Math. Phys. 46 (1976) 207.

[15] J. Tafel, Talk delivered at Dipartimento di Fisica, Università di Roma "La Sapienza" (1991) and at Université du Quebec a Montréal (1992).

[16] J. Tafel, Two-dimensional reductions of the self-dual Yang–Mills equations in self-dual spaces, J. Math. Phys. 34 (1993) 1892.

[17] R.S. Ward, *Multi-dimensional Integrable Systems*, Lecture Notes in Physics, Vol. 280 (Springer, Berlin, 1987).

[18] V.E. Zakharov and A.V. Mikhailov, Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique, Sov. Phys. JETP 47 (1978) 1017; Zh. Eksp. Teor. Fiz. 74 (1978) 1953.